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Smooth Interpolation in Triangles

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The purpose of this paper is to describe new schemes of interpolation to the boundary values of a function defined on a triangle. These schemes are affine-invariant and combine several Hermite interpolants. They are not, however, finite dimensional schemes. The simplest scheme is exact for quadratic functions, uses rational linear weighting ("blending") functions analogous to the methods of Mangeron and Coons for rectangles, and satisfies a maximum principle. For any positive integer p, there is an analogous scheme which interpolates on the boundary to the function and all its partial derivatives of order $p - 1$. The interpolant satisfies a partial differential equation of order $6p$ and approximates any sufficiently smooth function to order $O(h^{3p})$.

1. INTRODUCTION

The *interpolation problem* of constructing a smooth function of two or more variables which assumes given values on the boundary $\partial\Gamma$ of a given region Γ arises in many applications. So does the more general interpolation problem of constructing, for a given positive integer p, a smooth function defined over Γ having given values and normal derivatives $\partial^k u/\partial n^k$ for $k = 0,..., p - 1$ on $\partial\Gamma$.

For Γ a disc and $p = 1$, a satisfactory solution to this interpolation problem is given by the Poisson integral formula. The resulting *harmonic* *interpolant* is that unique function which assumes the given boundary values on $\partial\Gamma$ and satisfies $\nabla^2 u = 0$ inside Γ. For $p > 1$, there is a unique *polyharmonic interpolant* which satisfies $\nabla^{2p} u = 0$ inside Γ and has the appropriate normal derivatives on $\partial\Gamma$; this interpolant is given by an analogous integral formula [10].

For Γ a rectangle with sides parallel to the axes and $p = 1$, Mangeron [9] found some decades ago an even simpler construction. The interpolant which he constructed (and which is widely used by draftsmen in computer-aided design), he showed to be the unique solution of the differential equation $\partial^4 u/\partial x^2\, \partial y^2 = 0$ which assumes the given boundary values. A more direct interpretation of this solution was given several years ago by Coons [2], who also showed how to interpolate more generally, for any positive integer p, to the values and first $p - 1$ normal derivatives of a function given on the boundary ∂R of a rectangle R, provided that the specified derivatives are compatible at the corners and reasonably smooth. The resulting interpolation scheme, which is very simple computationally, was later shown by two of us [1] to give the unique solution of the differential equation $\partial^{4p} u/\partial x^{2p}\, \partial y^{2p} = 0$ for the prescribed boundary data. For any function $F \in C^{4p}(R)$, the order of accuracy of Coon's pth order scheme is $O(h^{4p})$ in a rectangle of diameter h.

In the present paper, we solve the corresponding problem for given (compatible) boundary values and derivatives on the edges of a triangle T. Our interpolating (or "blending") schemes are affinely invariant. Moreover, for any values of $p = 1, 2, 3, \ldots$, the interpolating function W interpolates

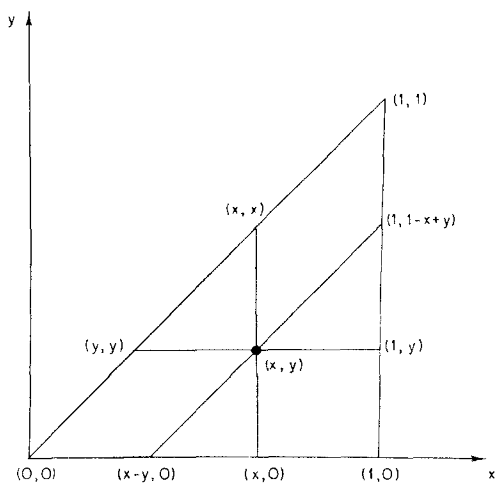

FIG. 1. Standard triangle.

to any $F \in C^{3p}(T)$ and to all of the first $p-1$ derivatives of F. The "blending functions" for these new interpolation schemes are rational functions which are bounded in T. The pth order scheme has order of accuracy $O(h^{3p})$, and the interpolant satisfies an appropriate partial differential equation of order $6p$.

For algebraic simplicity, we shall let T denote the "standard triangle" with vertices at $(0,0)$, $(1,0)$, and $(1,1)$ in the $(x-y)$-plane; see Fig. 1. Any other triangle can be obtained from this standard triangle by an affine transformation which carries polynomial and rational functions into polynomial and rational functions of the same degree, and preserves the order of approximation. For example, the *barycentric* (or "areal") coordinates $z_0 = 1 - x$, $z_1 = x - y$, $z_2 = y$ map it onto the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ in the (z_1, z_2)-plane, whose projective coordinates (z_0, z_1, z_2) are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively.

2. Semigroup of Projectors

For any continuous function F on T ($F \in C(T)$), consider the three *projectors* (idempotent linear operators) $\mathscr{P}_i : F \to \mathscr{P}_i[F] = U_i(x,y)$ defined by the formulas

$$U_1(x,y) = \left(\frac{1-x}{1-y}\right) F(y,y) + \left(\frac{x-y}{1-y}\right) F(1,y), \tag{1a}$$

$$U_2(x,y) = \left(\frac{x-y}{x}\right) F(x,0) + \left(\frac{y}{x}\right) F(x,x), \tag{1b}$$

$$U_3(x,y) = \left(\frac{1-x}{1-x+y}\right) F(x-y,0) + \left(\frac{y}{1-x+y}\right) F(1, 1-x+y). \tag{1c}$$

It is easy to check that each U_i represents simple linear interpolation along segments parallel to the ith side of T, between the values assumed by F on the other two sides. In other words, the graph $z = U_i(x,y)$ of each function $U_i = \mathscr{P}_i[F]$ is a ruled surface which interpolates to F between two of the three lines $y = 0$, $x = 1$, and $x = y$, by rulings whose projections on the x, y-plane are parallel to those of the third side. This description is evidently preserved under affine transformation.

We now consider the multiplicative semigroup which the $\mathscr{P}_i$ generate under left-composition. This is most easily determined by considering

$$C(T) = L(T) \oplus Z(T) \tag{2}$$

as the direct sum of the subspace $L(T)$ of all *linear* functions in x and y and the subspace $Z(T)$ of all functions $g(x, y)$ which vanish at the three corners of T. Evidently, each $\mathscr{P}_i$ acts like the identity on $L(T)$; hence so does every product of $\mathscr{P}_i$. Again, $\mathscr{P}_i$ annihilates every $g \in Z(T)$ on the ith side of ∂T, and interpolates linearly along parallels to this side between the values on the other two sides. Explicitly, for T the standard triangle, we have for example

$$\mathscr{P}_1\mathscr{P}_2[F] = \left(\frac{1-x}{1-y}\right) F(y, y) + \left(\frac{x-y}{1-y}\right) [(1-y) F(1, 0) + yF(1, 1)], \quad (3)$$

and

$$\mathscr{P}_2\mathscr{P}_1[F] = \left(\frac{x-y}{x}\right) [(1-x) F(0, 0) + xF(1, 0)] + \left(\frac{y}{x}\right) F(x, x), \quad (4)$$

which is not the same as $\mathscr{P}_1\mathscr{P}_2$. (This is in contrast to the case of rectangles treated in [1].)

It is interesting to note that even though the projections $\mathscr{P}_i\mathscr{P}_j[F]$ and $\mathscr{P}_j\mathscr{P}_i[F]$ $(i \neq j)$ are different bivariate functions, they do coincide on the boundary of T:

$$\mathscr{P}_i\mathscr{P}_j[F]|_{\partial T} = \mathscr{P}_j\mathscr{P}_i[F]|_{\partial T}. \quad (5)$$

From the previous results, we easily derive

$$\mathscr{P}_i^2 = \mathscr{P}_i, \qquad \mathscr{P}_i\mathscr{P}_j \neq \mathscr{P}_j\mathscr{P}_i, \qquad \mathscr{P}_i\mathscr{P}_j\mathscr{P}_i = \mathscr{P}_i\mathscr{P}_j,$$

while $\mathscr{P}_i\mathscr{P}_j\mathscr{P}_k$ $(i, j, k$ distinct) projects $Z(T)$ onto 0. Therefore, the semigroup generated by the projections contains ten elements, all of which are projections. (A similar construction can be made for tetrahedra, etc.)

Now consider the interpolation schemes defined by the six *quasi-Boolean* sums of projectors:

$$\mathscr{P}_i \oplus \mathscr{P}_j = \mathscr{P}_i + \mathscr{P}_j - \mathscr{P}_i\mathscr{P}_j : F(x, y) \to U_{ij}(x, y). \quad (6)$$

LEMMA 1. *The six quasi-Boolean sums defined by* (6) *all interpolate to F on ∂T.*

We omit the proof, which reduces to a straightforward computational verification of the identities

$$U_{ij}(x, 0) = F(x, 0), \qquad U_{ij}(1, y) = F(1, y), \qquad U_{ij}(y, y) = F(y, y) \quad (7)$$

for all $i \neq j$ with $i, j = 1, 2, 3$.

COROLLARY. *For all three pairs $\{i, j\}$ with $i \neq j$, the function*

$$V_{ij}(x, y) = V_{ji}(x, y) = \tfrac{1}{2}[U_{ij}(x, y) + U_{ji}(x, y)] \quad (8)$$

interpolates to F on ∂T.

We can, of course, define the V_{ij} directly by

$$V_{ij} = Q_{ij}[F], \qquad Q_{ij} = \mathcal{P}_i + \mathcal{P}_j - \tfrac{1}{2}[\mathcal{P}_i\mathcal{P}_j + \mathcal{P}_j\mathcal{P}_i]. \tag{9}$$

LEMMA 2. *The product $\mathcal{L} = \mathcal{P}_i\mathcal{P}_j\mathcal{P}_k$ ($i \neq j \neq k \neq i$), in any order, of the three projectors $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ defined by* (1a)–(1c) *is given by*

$$\mathcal{L}[F] = L, \tag{10}$$

where

$$L(x, y) = (1 - x)\,F(0, 0) + (x - y)\,F(1, 0) + yF(1, 1). \tag{10$'$}$$

In other words, the graph of $z = L(x, y)$ in x, y, z-space is the plane through the three vertices of the graph of $z = F(x, y)$.

3. TRILINEAR BLENDING

With the help of the formulas of Section 2, it is easy to describe our first symmetric interpolation scheme. We shall refer to this scheme as *trilinear blending*, since it is built up from the projectors $\mathcal{P}_i$, and each $\mathcal{P}_i$ interpolates linearly in x, y, z-space between parallels to the ith side of T. The scheme and some of its basic properties can be described as follows.

THEOREM 1. *Let Q^* be defined by*

$$Q^* = \tfrac{1}{2}[\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 - \mathcal{L}]. \tag{11}$$

Then Q^* is a projector on $C(T)$. Moreover, for any $F \in C(T)$, the function $W = Q^*[F]$ is given by

$$\begin{aligned} W(x, y) = \frac{1}{2}\Big\{&\Big[\Big(\frac{1 - x}{1 - y}\Big) F(y, y) + \Big(\frac{x - y}{1 - y}\Big) F(1, y)\Big] \\ &+ \Big[\Big(\frac{x - y}{x}\Big) F(x, 0) + \Big(\frac{y}{x}\Big) F(x, x)\Big] \\ &+ \Big[\Big(\frac{1 - x}{1 - x + y}\Big) F(x - y, 0) + \Big(\frac{y}{1 - x + y}\Big) F(1, 1 - x + y)\Big] \\ &- [(1 - x)\,F(0, 0) + (x - y)\,F(1, 0) + yF(1, 1)]\Big\}. \end{aligned} \tag{12}$$

Proof. Again, the proof is straightforward. Note that the rational weighting functions (blending functions) in (12) are continuous except at the corners of T. On the other hand, considered in pairs, the rational weighting functions sum to 1, and the functions being averaged (e.g., $F(y, y)$ and $F(1, y)$ near the corner $(1, 1)$) approach the same value. Hence, if $F \in C(T)$, then $W \in C(T)$.

Remark. Note the interesting identities:

$$Q^* = \tfrac{2}{3} \sum_{i=1}^{3} \mathscr{P}_i - \tfrac{1}{6} \sum_{i \neq j} \mathscr{P}_i \mathscr{P}_j = \tfrac{1}{6} \sum_{i \neq j} \mathscr{P}_i \oplus \mathscr{P}_j, \tag{13}$$

$$Q^* = \tfrac{1}{2}[(\mathscr{P}_i \oplus \mathscr{P}_j) + (\mathscr{P}_i \oplus \mathscr{P}_k)] \qquad (i \neq j \neq k \neq i). \tag{14}$$

Note that $\mathscr{P}_i \mathscr{L} = \mathscr{L} \mathscr{P}_i = \mathscr{L}$ for any $i = 1, 2, 3$. From this and (11), we obtain the equation

$$Q^* = \tfrac{1}{2}[(\mathscr{P}_1 + \mathscr{P}_2 + \mathscr{P}_3)(I - \mathscr{L})] + \mathscr{L}, \tag{15}$$

where I is the identity operator. From (15), we can interpret the construction of $W = Q^*[F]$ as follows: First, pass a plane in x, y, z-space through the corners of T, the graph of $z = F(x, y)$, and reduce to the graph of the function $F - \mathscr{L}[F]$ which has zero corner values. Next interpolate linearly to $F - \mathscr{L}[F]$ between each of the three pairs of sides and take half the sum of the functions whose graphs are these three ruled surfaces. Then,

$$W = \mathscr{L}[F] + \tfrac{1}{2}\{(\mathscr{P}_1 + \mathscr{P}_2 + \mathscr{P}_3)[F - \mathscr{L}[F]]\}. \tag{16}$$

COROLLARY (Maximum Principle). *The interpolant $W(x, y)$ of Theorem 1 satisfies*

$$\max_T |W(x, y)| \leqslant 2 \max_{\partial T} |F(x, y)|. \tag{17}$$

If $F = 0$ at the corners of T, the factor 2 can be replaced by 3/2.[1]

Proof. Consider (11), and note that

$$\max_T |\mathscr{P}_i[F]| \leqslant \max_{\partial T} |F| \qquad (i = 1, 2, 3),$$

and

$$\max_T |\mathscr{L}[F]| \leqslant \max_{\partial T} |F|.$$

Use the triangle inequality to obtain (17). If $F = 0$ at the corners of T, then $\mathscr{L}[F] = 0$, in which case the factor 3/2 obtains.

[1] In formula (17) of [1], a similar maximum principle is given for bilinearly blended interpolation over the unit square. The bound given there, namely, $\max_S |U| \leqslant 2 \max_{\partial S} |F|$, is valid only if $F = 0$ at the four corners of S. In general, the factor 2 must be replaced by 3.

THEOREM 2. *Let F be of class $C^6(T)$, so that the boundary values of F are of class C^6. Then the interpolant W of Theorem 1 satisfies the sixth order partial differential equation*

$$\left[\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\right]^2 W(x, y) = E^2[W] = 0. \tag{18}$$

Proof. The result is essentially a consequence of the observation that, in (1a – c) $\partial^2 U_1/\partial x^2 = 0$, $\partial^2 U_2/\partial y^2 = 0$, $\partial^2 U_3/\partial(x - y)^2 = 0$, and (11). More precisely, $\partial^3/\partial x\, \partial y\, \partial(x - y)$ acting on any term of (11) gives a function $E[U_i]$ which is of the first degree in one of x, y and $x - y$ and, hence, satisfies $E^2[U_i] = 0$.

Remark. More generally, for any $F \in C^5(T)$, if the sum $(\partial/\partial x + \partial/\partial y)$ is interpreted as the vector derivative with respect to the vector $(1, 1)$, the analog of (18) holds. This is obvious since the fifth derivative

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left[\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}+\frac{\partial W}{\partial y}\right)\right]$$

then exists and is the sum of two zero terms and a function which is linear in x for each fixed y. (Clearly, the six differential operators can be applied in any order, giving 90 variants of (18) valid for any $F \in C^5(T)$).

4. REMAINDER THEORY

We easily verify that the trilinear interpolation scheme of Section 3 is exact for linear and quadratic functions. (Try $x(1 - x)$, and observe that the exactness of Q^* for $x(1 - x)$ and invariance under affine transformation implies the exactness of Q^* for $(1 - y)y$ and for $(x - y)(1 - x + y)$.) It is not exact for all cubic polynomial functions; thus, $Q^*[(1 - x)y(x - y)] \neq 0$. This suggests that the error in trilinear interpolation is $O(h^3)$ for a triangle of diameter h, a result which we now prove as follows.

First, since trilinear interpolation is exact for linear functions, we can assume that F vanishes at the corners. In this case, the function $W(x, y)$ in (12) is the sum of six terms, each of which: (i) vanishes at the corners, (ii) is defined in an affinely invariant way, and (iii) is equivalent to every other term under the group of (six) affine transformations induced by permuting the vertices. Hence, it suffices to consider in detail a single summand in (12), since the class $C^n(T)$ is also invariant under affine transformation; we choose $yF(x, x)/x$. Since $F(0, 0) = 0$, we can use the following lemma.

LEMMA 3. *If* $f(x) \in C^{n+1}[0, x]$ *and* $f(0) = 0$, *then* $g(x) = f(x)/x \in C^n[0, x]$.

Proof. Using Taylor's theorem with remainder in integral form, an elementary calculation gives

$$g(x) = \sum_{l=0}^{n} f^{(l+1)}(0)\, x^l/(l+1)! + \int_0^x f^{(n+1)}(t) \left(\frac{x-t}{x}\right) \frac{(x-t)^{n-1}}{(n!)}\, dt. \quad (19a)$$

We can differentiate n times with respect to x under the integral sign; since $|(x-t)/x| \leq 1$, the resulting integral will tend to zero as $x \to 0$. (However, the functions x^ϵ and $x^{1+\epsilon}\sin(1/x)$ show that one must assume $f \in C^{n+1}$.)

More in detail, differentiating this series n times by Leibniz' rule (see [8, p. 219]), we get

$$g^{(n)}(x) = f^{(n+1)}(0)/(n+1) + \int_0^x f^{(n+1)}(x)\, G(x, t)\, dt,$$

where

$$G(x, t) = \frac{1}{n!} \frac{d^n}{dx^n} \left[\left(1 - \frac{t}{x}\right)(x-t)^{n-1}\right].$$

But since $d^n(uv)/dx^n = \sum \binom{n}{k} u^{(n-k)} v^{(k)}$:

$$G(x, t) = \sum_{k=1}^{n} \binom{n}{k} \left[(-1)^k (k!) \frac{t(x-t)^{k-1}}{x^{k+1}(n!)}\right]$$

$$= \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)!} \left(\frac{t}{x}\right)\left(1 - \frac{t}{x}\right)^{k-1} \cdot \frac{1}{x}.$$

Hence, setting $\tau = t/x$, we have

$$g^{(n)}(x) = \frac{f^{(n+1)}(0)}{n+1} + \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)!} \int_0^1 \tau(1-\tau)^{k-1} f^{(n+1)}(\tau)\, d\tau.$$

Applying the second law of the mean to each term of the final sum, we get

$$\| g^{(n)}(x) \|_\infty \leqslant K_n \| f^{(n+1)} \|_\infty, \quad (19b)$$

where

$$K_n = \frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{(n-k)!} \int_0^1 \sigma^{k-1}(1-\sigma)\, d\sigma, \qquad (\sigma = 1 - \tau).$$

The last integral is $1/k(k+1)$; hence,

$$K_n = \frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{k(k+1)(n-k)!}. \quad (19c)$$

It is important to show that the $U_i = \mathscr{P}_i[F]$ are *uniformly* smooth. By affine similarity, it suffices to consider the case of the vertex at (0, 0) of the "unit triangle" with vertices at (0, 0), (1, 0), and (1, 1). Moreover, without affecting any derivatives of order two or more, we can assume that $F(0, 0) = F(1, 0) = F(1, 1) = 0$; this we do. Accordingly, we consider

$$U_2(x, y) = F(x, 0) + yG(x), \qquad G(x) = [F(x, x) - F(x, 0)]/x.$$

Setting $\eta = y/x$, we have

$$G(x) = \frac{1}{x} \int_0^x \frac{\partial F}{\partial y}(x, t)\, dt = \int_0^1 \frac{\partial F}{\partial y}(x, x\eta)\, d\eta.$$

If $F \in C^{n+1}(T)$, it follows by Leibniz' rule (see [8, p. 219]) that $G \in C^n[0, 1]$. Hence, $U_2(x, y) = F(x, 0) + yG(x) \in C^n(T)$. Although $G^{(n+1)}(x)$, and, hence, $\partial^{n+1} U_2/\partial x^{n+1}$ need not exist.

We now apply the preceding results to the error (remainder)

$$F(x, y) - W(x, y) = R(x, y).$$

This is in $C^3(T)$ if $F \in C^4(T)$; moreover, the kth partial derivatives of W are in bounded ratio to those of F; hence, the same is true of those of R.

THEOREM 3. *If $F \in C^4(T)$, then the error $R = F - W$ is $O(h^3)$, where h is the diameter of T.*[2]

Proof. Since $R \in C^3(T)$ vanishes on $x = 1$, by Lemma 3 $R = (1 - x)R_1(x, y)$, where $R_1 \in C^2(T)$ vanishes on $y = 0$ and $x = y$. By Lemma 3 again, $R = (1 - x)\, yR_2(x, y)$, where $R_2 \in C^1(T)$. Applying the same reasoning a third time, we have

$$F(x, y) = W(x, y) + (1 - x)\, y(x - y)\, S(x, y), \tag{20}$$

where S is continuous and bounded (indeed, uniformly bounded in terms of the maximum third derivative of F).

5. TRICUBIC BLENDING

We shall now show how to interpolate to boundary values and normal[3] derivatives of smooth functions in triangles. We first consider the case of *cubic* blending functions, $p = 2$.

[2] In a forthcoming paper entitled "Error Bounds for Smooth Interpolation in Triangles," R. E. Barnhill and Lois Mansfield provide alternative proofs for the error bounds given in the present paper. Their proofs are based upon the Sard kernel theorem.

[3] Formulas which interpolate to boundary values and to *normal* derivatives of orders $1, \ldots, p - 1$ automatically interpolate to *all* partial derivatives of orders $1, \ldots, p - 1$, since these are tangential derivatives of normal derivatives.

As in Section 2, we begin with three projectors $\tilde{\mathscr{P}}_1$ ($i = 1, 2, 3$). Each $\tilde{\mathscr{P}}_i$ replaces any $F(x, y) \in C^1(T)$ by its *cubic Hermite* interpolant along parallels to the ith side. Thus, for the projector $\tilde{\mathscr{P}}_1$, set

$$X = (x - y)/(1 - y), \tag{21}$$

so that X ranges from 0 to 1 on any segment parallel to the side $y = 0$ between the sides $x = y$ and $x = 1$. Then define

$$\tilde{\mathscr{P}}_1 : F \to \tilde{\mathscr{P}}_1[F] \tag{22}$$

as follows:

$$\begin{aligned} \tilde{\mathscr{P}}_1[F] = {} & \phi_1(X)\, F(y, y) + \phi_2(X)(1 - y)\, F_x(y, y) \\ & + \phi_3(X)\, F(1, y) + \phi_4(X)(1 - y)\, F_x(1, y), \end{aligned} \tag{23a}$$

where X is given by (21) and

$$\begin{aligned} \phi_1(X) &= X^2(2X - 3) + 1, & \phi_2(X) &= X(X - 1)^2, \\ \phi_3(X) &= -X^2(2X - 3), & \phi_4(X) &= X^2(X - 1), \end{aligned} \tag{23b}$$

are the "Hermite cardinal functions" for interpolation over [0, 1].

For $\tilde{\mathscr{P}}_2$ and $\tilde{\mathscr{P}}_3$ similar formulas hold, except that (21) is replaced by

$$X = \frac{y}{x} \quad \text{if} \quad i = 2, \qquad \text{and} \qquad X = \frac{y}{1 - x + y} \quad \text{if} \quad i = 3.$$

Explicitly, we have the projectors

$$\begin{aligned} \tilde{\mathscr{P}}_2[F] = {} & \frac{(y - x)^2\,(x + 2y)}{x^3}\, F(x, 0) + \frac{(y - x)^2\, y}{x^2}\, F_y(x, 0) \\ & + \frac{y^2(3x - 2y)}{x^3}\, F(x, x) + \frac{y^2(y - x)}{x^2}\, F_y(x, x) \end{aligned} \tag{24}$$

$$\begin{aligned} \tilde{\mathscr{P}}_3[F] = {} & \frac{(1 - x)^2\,(3y - x + 1)}{(1 - x + y)^3}\, F(x - y, 0) \\ & + \frac{(x - 1)^2\, y}{(1 - x + y)^2}\,[F_x(x - y, 0) + F_y(x - y, 0)] \\ & + \frac{y^2(-3x + y + 3)}{(1 - x + y)^3}\, F(1, 1 - x + y) \\ & + \frac{y^2(x - 1)}{(1 - x + y)^2}\,[F_x(1, 1 - x + y) + F_y(1, 1 - x + y)]. \end{aligned} \tag{25}$$

These projectors have algebraic properties similar to those of the simpler projectors studied in Section 2. For example, we have the following higher-order extension of Lemma 1.

LEMMA 4. *For $i, j = 1, 2, 3$, and $i \neq j$, the functions $\tilde{U}_{ij} = (\tilde{\mathscr{P}}_i \oplus \tilde{\mathscr{P}}_j)[F]$ interpolate to $F \in C^2(\bar{T})$ and its first-order normal derivative of $\partial F/\partial n$ on ∂T:*

$$\tilde{U}_{ij}\Big|_{\partial T} = F\Big|_{\partial T} \qquad \textit{and} \qquad \frac{\partial \tilde{U}_{ij}}{\partial n} = \frac{\partial F}{\partial n}\Big|_{\partial T}. \tag{26}$$

Proof. Consider the function $\tilde{U}_{12} = \tilde{\mathscr{P}}_1 \oplus \tilde{\mathscr{P}}_2$

$$\tilde{U}_{12}(x, y) = \tilde{\mathscr{P}}_1[F] + \tilde{\mathscr{P}}_2[F] - \tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F]. \tag{27}$$

On the sides of the triangle T, the function $\tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F]$ has the values

$$\tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F] = \begin{cases} F(y, y) & \text{on } x = y, \\ F(0, 0)\,\phi_1(x) + F_y(0, 0)\,\phi_2(x) + F(1, 0)\,\phi_3(x) + F_y(1, 0)\,\phi_4(x) \\ \qquad\qquad \text{on } y = 0, \\ F(1, 0)\,\phi_1(y) + F_y(1, 0)\,\phi_2(y) + F(1, 1)\,\phi_3(y) + F_y(1, 1)\,\phi_4(y) \\ \qquad\qquad \text{on } x = 1. \end{cases} \tag{28}$$

where the functions ϕ_i are the cubic Hermite cardinal functions of (23b). Moreover, along the two sides $y = x$ and $x = 1$, the first-order directional derivative (in any direction) of the function $\tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F]$ coincides with that of $\tilde{\mathscr{P}}_2[F]$. In particular, the directional derivatives of $\tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F]$ and $\tilde{\mathscr{P}}_2[F]$ match those of F along $y = x$. Along the remaining side, $y = 0$, we have $(\partial/\partial y)\,\tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F] = (\partial/\partial y)\,\tilde{\mathscr{P}}_1[F]$. With these facts in mind, it is easy to verify (26). For example, along $y = 0$ we have

$$\tilde{U}_{12}(x, 0) = \tilde{\mathscr{P}}_1[F]\Big|_{y=0} + \tilde{\mathscr{P}}_2[F]\Big|_{y=0} - \tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F]\Big|_{y=0}$$

which, since $\tilde{\mathscr{P}}_1[F] = \tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F]$ and $\tilde{\mathscr{P}}_2[F] = F(x, 0)$ on $y = 0$, gives $\tilde{U}_{12}(x, 0) = F(x, 0)$. Similarly,

$$\frac{\partial \tilde{U}_{12}(x, 0)}{\partial y} = \left\{\frac{\partial}{\partial y}\tilde{\mathscr{P}}_1[F] + \frac{\partial}{\partial y}\tilde{\mathscr{P}}_2[F] - \frac{\partial}{\partial y}\tilde{\mathscr{P}}_1\tilde{\mathscr{P}}_2[F]\right\}\Big|_{y=0}.$$

From the foregoing arguments, we note that the first and third of the terms on the right cancel and that the second is equal to $F_y(x, 0)$ because of the interpolatory properties of $\tilde{\mathscr{P}}_2$. Analogous considerations serve to establish interpolation to F and its normal derivative along the remaining two sides of T. The other five possible cases for $\tilde{U}_{ij}$ follow by affine invariance and symmetry.

Since each of the six functions $\tilde{U}_{ij}$ $(i \neq j)$ in Lemma 4 interpolates to F and its first-order normal derivative along the boundary of T, we have the following symmetric tricubic blending scheme which is the analog of the trilinear blending scheme of Theorem 1.

THEOREM 4. *The function* $\tilde{W} = \tilde{Q}[F]$, *where*

$$\tilde{Q} = \tfrac{2}{3} \sum_{i=1}^{3} \tilde{\mathscr{P}}_i - \tfrac{1}{6} \sum_{i \neq j} \tilde{\mathscr{P}}_i \tilde{\mathscr{P}}_j , \tag{29}$$

and the $\tilde{\mathscr{P}}_i$ *are given by* (23), (24), *and* (25), *satisfies*

$$\tilde{W} = F \quad \text{and} \quad \frac{\partial \tilde{W}}{\partial n} = \frac{\partial F}{\partial n} \quad \text{on } \partial T. \tag{30}$$

The other results of Sections 3 and 4 also have analogs for the tricubic blending scheme of (29). For example, we have the following analog of Theorem 2; its proof is similar.

THEOREM 5. *If* $F \in C^{n+2}(T)$, *then* $\tilde{W} = \tilde{Q}[F] \in C^n(T)$ *and, if* $F \in C^{12}(T)$ *then*

$$\left[\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\right]^4 \tilde{W}(x, y) = E^4[\tilde{W}] = 0 \quad \text{in } T. \tag{31}$$

Likewise, we have a straightforward analog of Theorem 3, again with a similar proof.

THEOREM 6. *If* $F \in C^9(T)$, *then the error* $\tilde{R} = F - \tilde{W}$ *in tricubic blending is* $O(h^6)$, *where* h *is the diameter of* T.

As a corollary of Theorem 6, we have

$$F(x, y) = \tilde{W}(x, y) + [xy(1 - x - y)]^2 \tilde{H}(x, y), \qquad \tilde{H} \in C(T) \tag{32}$$

for all $F \in C^9(T)$.

6. Higher-Order Interpolation

The generalization of trilinear and tricubic blending in triangles to any positive integer p is straightforward. Thus, for $p = 3, 4, 5, \ldots$, define the three projectors $\tilde{\mathscr{P}}_1{}^p$ to be the Hermite interpolants to the values and first $p - 1$ (directional) derivatives (parallel to the ith side) on two sides of the triangle T, interpolated along parallels to the third side. For example, the function $U_1{}^p = \mathscr{P}_1{}^p[F]$ is constructed by using the cardinal polynomials $\phi_1(X), \phi_2(X), \ldots, \phi_{2p}(X)$, with $X = (x - y)/(1 - y)$ as in (21), defined for Hermite interpolation between $X = 0$ and $X = 1$ as in [3, p. 37]. Thus, for the interval $[y, 1]$ with y fixed, we have

$$U_1{}^p(x, y) = \sum_{i=1}^{p} \{\phi_i(X)(1 - y)^{i-1} F^{(i-1,0)}(y, y) + \phi_{p+i}(X)(1 - y)^{i-1} F^{(i-1,0)}(1, y)\}. \tag{33}$$

By the known properties of Hermite interpolation,

$$\partial^\mu U_1{}^p/\partial x^\mu = \partial^\mu F/\partial x^\mu$$

along the sides $x = y$ and $x = 1$ for $\mu = 0, 1, \ldots, p - 1$.

For $F \in C^{4p}(T)$, the error in the preceding interpolation scheme is $O(h^{3p})$, and for $F \in C^{3p}(T)$ the interpolating function satisfies

$$\left[\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\right]^{2p} W = 0.$$

Affine Invariance

The formulas of Sections 2–6 provide triangular analogs of the rectangular formulas considered in [1], [2], and [6]. These triangular schemes have the geometrically appealing property of being affinely invariant, because to interpolate to F and its normal derivatives of orders $k = 1, \ldots, p - 1$ is equivalent to interpolating to F and all its partial derivatives of orders $k = 1, \ldots, p - 1$, and this is affinely invariant.

7. Other Interpolation Schemes

Although the main purpose of this paper has been the derivations and error analyses for the class of schemes in Sections 1–6, we conclude by constructing other formulas which also interpolate to the boundary values of F on ∂T.

In Sections 2 and 3, we observed that the three elementary projectors $\mathscr{P}_1$, $\mathscr{P}_2$, and $\mathscr{P}_3$ generate ten functions—namely, U_{ij} ($i, j = 1, 2, 3$, with $i \neq j$), V_{ij} ($i < j$, with $i = 1, 2$) and W—all of which interpolate to an arbitrary function F on ∂T. Moreover, any convex linear combination of these ten functions will also provide an interpolating function. Clearly, the difference between any two such functions is a *nontrivial* function which vanishes on ∂T. Such functions are potentially useful for surface design as "correction displacements," since they alter the shape of a surface $z = U(x, y)$ in the interior of T without affecting the boundary values.

We shall now show that by considering other projectors, it is easy to derive still other functions which solve the same interpolation problem as that of Section 3. For instance, by slightly altering the definitions of the projectors (1a) and (1b) to

$$\mathscr{P}_1[F] = \left(\frac{1-x}{1-y}\right) F(x, x) + \left(\frac{x-y}{1-y}\right) F(1, y), \tag{34a}$$

$$\mathscr{P}_2[F] = \left(\frac{x-y}{x}\right) F(x, 0) + (y/x)\, F(y, y), \tag{34b}$$

we obtain the two interpolants $Z_1 = (\mathscr{P}_1 \oplus \mathscr{P}_2)[F]$ and $Z_2 = (\mathscr{P}_2 \oplus \mathscr{P}_1)[F]$:

$$\begin{aligned} Z_1(x, y) = {} & \left(\frac{x-y}{x}\right) F(x, 0) + \left(\frac{x-y}{1-y}\right) F(1, y) \\ & + (y-x)\, F(1, 0) + \left(\frac{y(1-x)(1+x-y)}{x(1-y)}\right) F(y, y), \end{aligned} \tag{35a}$$

$$\begin{aligned} Z_2(x, y) = {} & \left(\frac{x-y}{x}\right) F(x, 0) + \left(\frac{x-y}{1-y}\right) F(1, y) \\ & + (y-x)\, F(1, 0) + \left(\frac{y(1-x)(1+x-y)}{x(1-y)}\right) F(x, x). \end{aligned} \tag{35b}$$

The reader can easily confirm that $Z_1 = Z_2 = F$ on ∂T, and that $\mathscr{P}_1\mathscr{P}_2 \neq \mathscr{P}_2\mathscr{P}_1$.

However, the projectors $\mathscr{P}_1'$ and $\mathscr{P}_2'$ defined by

$$\mathscr{P}_1'[F] = \left(\frac{x-y}{x}\right) F(x, 0), \tag{36a}$$

$$\mathscr{P}_2'[F] = \left(\frac{x-y}{1-y}\right) F(1, y) + \left(\frac{1-x}{1-y}\right) (y/x)\, F(x, x) \tag{36b}$$

do commute. Their Boolean sum $\mathscr{P}_1' \oplus \mathscr{P}_2'$ gives a function

$$\begin{aligned} Z_3(x, y) &= (\mathscr{P}_1' \oplus \mathscr{P}_2')[F] \\ &= \left(\frac{x-y}{x}\right) F(x, 0) + \left(\frac{x-y}{1-y}\right) F(1, y) \\ &\quad + (y-x)\, F(1, 0) + \left(\frac{1-x}{1-y}\right) (y/x)\, F(x, x), \end{aligned} \tag{37}$$

which differs from any which have been previously derived, but it satisfies the same interpolation conditions, viz., $Z_3 = F$ on ∂T. (See [4, pp. 250–251].)

Finally, we consider another set of two commutative projectors $\mathscr{P}_1''$ and $\mathscr{P}_2''$

$$\mathscr{P}_1''[F] = xF(1, y/x), \tag{38a}$$

$$\mathscr{P}_2''[F] = \left(\frac{x-y}{x}\right) F(x, 0) + (y/x) F(x, x). \tag{38b}$$

From these, we obtain the formula $Z_4 = (\mathscr{P}_1'' \oplus \mathscr{P}_2'')[F]$:

$$\begin{aligned} Z_4(x, y) = \left(\frac{x-y}{x}\right) F(x, 0) + xF(1, y/x) \\ + (y - x) F(1, 0) - yF(1, 1) + (y/x) F(x, x). \end{aligned} \tag{39}$$

The reader can easily confirm that $Z_4 = F$ on ∂T.

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